The critical exponent v of directed TSAWs and TSALFs on Sierpinski carpets: an analytic approach

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COMMENT

The critical exponent $\nu_\perp$ of directed TSAWs and TSALFs on Sierpinski carpets: an analytic approach

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Abstract. It is shown rigorously that the critical exponent $\nu_\perp$ for a directed true self-avoiding walk on an arbitrary self-similar carpet is 1. Hence the simulation values by Yao *et al* represent an artefact due to finite-size effects. The corresponding exponent for a Lévy flight with parameter u is $\max(1, 1/u)$.

The directed true self-avoiding flight with parameter u on a carpet is defined by a probability distribution $p(r) \sim r^{-u-1}$ for the step length $r \in \mathbb{N}$ as follows. Let $\boldsymbol{F}$ be the position of the flyer, $\boldsymbol{x}$ and $\boldsymbol{y}$ the unit vectors in direction x and y, respectively. $P = \Sigma_{(h)}\, p(r)/2$ is the *a priori* probability that the flyer will arrive after the next step in a hole of the carpet. Hence the index (h) means that one has to sum over all r with $\boldsymbol{F} + r\boldsymbol{x}$ or $\boldsymbol{F} + r\boldsymbol{y}$ lying in a hole. (If both end-points lie in a hole then $p(r)/2$ is counted twice in the sum.) Now, if $r\boldsymbol{x}$ or $r\boldsymbol{y}$ is an admissible step it has the *a posteriori* probability $p(r)/(2-2P)$ [1, 2]. In the limit $u \to \infty$ we get a directed walk, i.e. the step length is always 1. A Lévy flight is characterized by the divergence of the variance of the step length, which is equivalent to $u \leqslant 2$. Since a directed walk is automatically self-avoiding it seems that the difference between the true self-avoiding walk (TSAW) and the usual self-avoiding walk (SAW) disappears. But the different weights in the probability distribution [2] can influence the critical parameters. The average is performed for the TSAW over the starting points; every admissible point has the same probability, whereas for the SAW every configuration has the same probability. The two critical exponents ν and $\nu_\parallel$ are independent of the kind of self-avoiding walk. They are defined as the averaged end-to-end displacement $\overline{R_*^2}^{1/2} \sim N^{\nu_*}$ (* denotes $\parallel$ or no index) and can be derived for both directed self-avoiding walks on an arbitrary carpet easily:

$$
\begin{aligned}
&R_\parallel = \sqrt{2}N/2 \to \nu_\parallel = 1 \\
&R_\parallel \leqslant R \leqslant N \to \nu = 1.
\end{aligned}
\tag{1}
$$

The exponent ν lies only slightly outside the intervals calculated by Yao *et al* [1]. Their claimed dependence of ν on the fractal dimension (cf the abstract in [1]) obviously does not appear. But the deviations for $\nu_\perp$ of the DTSAW are remarkable, e.g. for their first carpet they published the estimate $\nu_\perp = 0.59 \pm 0.01$. Let us derive for this carpet a lower bound of the form $\overline{R_\perp^2}^{1/2} \geqslant c_1 N$ which implies immediately $\nu_\perp = 1$. It will be

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evident how to get similar bounds with constants $c_2 > c_1$, $c_3 > c_1$ for their other two lattices. Incidentally, for the DSAW on a regular lattice there was published earlier an estimate 0.86 for ν [3] which was corrected by analytic considerations [4-6] to $\nu = 1$.

For notational simplicity, let us assume that N is a multiple of 4. Determine the integer n by $5^{n-1} < 3N/4 \leqslant 5^n$ and consider the hole $ABCD$ of size $5^n \times 5^n$ of an $(n+1)$-stage cell. Define the hexagon $\mathbb{H} = AS_1S_2S_3S_4S_5$ by $S_1 \in AD$, $S_5 = AB$, $\overline{S_1D} = \overline{S_5B} = 3N/4$, $S_1S_2 \| S_3S_4 \| AB$, $S_2S_3 \| S_4S_5 \| AD$ and $\overline{S_1S_2} = \overline{S_4S_5} = N/4$ (cf figure 1). The area of $\mathbb{H}$ is $(N/4 + 2 \times 5^n - 3N/2)N/4$, the area of the $(n+1)$-stage cell without the hole $ABCD$ is 24×5^{2n}. Therefore, the probability that a random initial point of the $(n+1)$-stage cell lies in $\mathbb{H}$ is at least 1/216. The hexagon $\mathbb{H}$ was constructed in such a way that every path of length N starting in an arbitrary point $P \in \mathbb{H}$ fulfils

$$R_\perp \geqslant \sqrt{2}N/4 \tag{4}$$

that is

$$\overline{R_\perp^2}^{1/2} > \sqrt{3}N/72. \tag{3}$$

Lower bounds sharper than (3) are available if $(n+k)$-stage cells, $k > 1$, are considered. Then the surroundings of the left lower corner of the holes of size $5^{n+1}, \ldots, 5^{n+k-1}$ contain additional points whose paths beginning there obey (2) definitely.

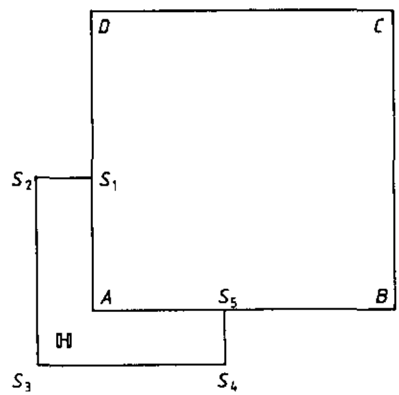

Figure 1. The hexagon $\mathbb{H}$ belonging to the hole $ABCD$ of the $(n+1)$-stage cell.

For $N = 2^7$, the inequality (3) yields, e.g., $\log_2 \overline{R_\perp^2}^{1/2} > 1.62$, which is still plainly below the value 3.403 shown in figure 6 of [1]. How must the proof be altered for partially directed TSAWs [7] with three possible directions (up, down, right) of the walker and directed true self-avoiding Lévy flights (DTSALF) [7, 8]? For the partially DTSAW, $\mathbb{H}$ becomes a rectangle of size $N/6 \times (W - N)$ around the centre of the left side of a hole with width W, whereas for the DTSALF with $u > 1$ the hexagon and the hole from figure 1 have to be multiplied by the expectation value $\langle r \rangle = \Sigma\, rp(r)$ of the step length. Equation (2) or its modification $R_\perp \geqslant \sqrt{2}N\langle r \rangle/4$ for the DTSALF is no longer valid for every path starting in $\mathbb{H}$, but only for a set with probability larger than some N-independent constant. Hence, $\nu_\perp = 1$ remains true in both cases as well as $\nu = \nu_\| = 1$, though instead of (1) only trivial upper bounds $R_\| \leqslant R \leqslant N$ for the partially DTSALF are fulfilled for every path. Nevertheless, the averages are still proportional to N. The arguments of [7] yield $\nu = \nu_\| = \nu_\perp = 1/u$ for the DTSALF with $u \leqslant 1$, i.e. the steps are so large that the holes of the fractal do not alter even the exponent $\nu_\perp$. These exponents differ again remarkably from the simulation values in [8].

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